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Bifurcation of Periodic Solutions*

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1. INTRODUCTION

Mechanical and electrical phenomena which can be described mathematically by the bifurcation or appearance of periodic solutions of a nonlinear ordinary different equation when some parameter is varied have been well-known for many years. See Minorsky [7]. In recent years, it has been observed that a number of biological and chemical phenomena can be described by bifurcation of periodic solutions and the Hopf Bifurcation Theorem (see Hopf [4]) has been widely applied. See, for examples, Hsü and Kazarinoff [5], Othmer [8], Othmer and Tyson [9], Poore [11], Troy [12]. For an extensive discussion of the Hopf Theorem and applications in fluid mechanics, see Marsden and McCracken [6].

The primary purpose of this paper is to describe extensions of the Hopf Bifurcation Theorem that are obtained by applying a general bifurcation theorem proved in an earlier paper [3]. The approach used is finite-dimensional: it is based on the classical approach of Poincaré [10], techniques introduced by Coddington and Levinson [2], and the use of topological degree. Roughly speaking, the result says that if certain smoothness conditions are satisfied and if the higher order term satisfies a simply stated nonzero condition then bifurcation occurs. The usual hypotheses about the behavior of the eigenvalues are largely avoided, but the condition on the higher order term is essential. As might be expected since degree theory is used, only existence results are obtained, i.e., we do not obtain continuous families of solutions.

In Section 2, the general bifurcation problem is described. In Section 3, we outline a finite-dimensional proof of the Hopf Bifurcation Theorem and then combine the technique of this proof with the bifurcation theorem in [3] to obtain extensions of the Hopf Bifurcation Theorem. Actually we show how the extension can be made in a specific case: the case in which

$$A(0) = \begin{bmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & D \end{bmatrix}$$

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where D is an $(n-3) \times (n-3)$ matrix which has no eigenvalues of the form qi where $q = 0, \pm 1, \ldots$. From this it will be clear how extensions can be made to cases which are more complicated but not fundamentally more difficult. In Section 4, we consider a slightly more general equation than that dealt with in the Hopf Bifurcation Theorem. To this we apply directly the technique of Coddington and Levinson [2] and then use the bifurcation theorem in [3] to obtain bifurcation of periodic solutions.

2. Bifurcation of Periodic Solutions in the General Case

In the general case, the bifurcation problem can be stated as follows: given the n-dimensional autonomous system:

$$x' = f(x, \epsilon) \tag{1}$$

where $x \in R^n$, and ϵ is real, and f has continuous second derivatives in all variables. Suppose that if $\epsilon = 0$, equation (1) has a nontrivial periodic solution $x_0(t)$ of period T_0. The problem is: does there exist a continuous function $T(\epsilon)$ such that $T(0) = T_0$ and a solution $x(t, \epsilon)$ of (1) with period $T(\epsilon)$ such that

$$\lim_{\epsilon \to 0} | x(t, \epsilon) - x_0(t)| = 0$$

uniformly in t? The standard approach to this question is to study the linear variational equation of (1) relative to the given solution $x_0(t)$, i.e., the equation

$$u' = \{f_x[x_0(t), 0]\}u. \tag{2}$$

Since $x_0(t)$ has period T_0, then $f_x[x_0(t), 0]$ is a matrix of period T_0 and system (2) has one characteristic multiplier equal to one. The classical result is that if the number one is a simple characteristic multiplier of (2) then for sufficiently small $| \epsilon |$, there is a continuous function $T(\epsilon)$ and a solution $x(t, \epsilon)$ of (1) with period $T(\epsilon)$ such that

$$\lim_{\epsilon \to 0} | x(t, \epsilon) - x_0(t)| = 0$$

uniformly in t. (See Coddington and Levinson [2].) If system (2) has more than one characteristic multiplier equal to one, the problem becomes more complicated. For example, if the given periodic solution $x_0(t)$ is nontrivial and a periodic solution of the form

$$x_0(t) + \epsilon\, v(t)$$

is sought, then the problem can become the study of periodic solutions of a fairly complicated non-autonomous system. However if we restrict ourselves to the case in which $x_0(t)$ is an equilibrium point, then the problem can be handled and fairly extensive results obtained. If $x_0(t)$ is an equilibrium point, then by translating the coordinate system, we may write equation (1) as

$$x' = A(\epsilon)x + f(x, \epsilon) + g(\epsilon) \tag{3}$$

where $A(\epsilon)$ is a differentiable $n \times n$ matrix; $g(\epsilon)$ is a differentiable function of ϵ such that $g(0) = 0$; and $f(x, \epsilon)$ is a differentiable function of (x, ϵ) such that

$$|f(x, \epsilon)| = o(|x|)$$

uniformly in ϵ for $|\epsilon|$ sufficiently small.

3. The Hopf Bifurcation Theorem and a Direct Extension of It

We assume that in equation (3), the function $g(\epsilon)$ is identically zero, i.e., we consider the equation

$$x' = A(\epsilon)x + f(x, \epsilon). \tag{4}$$

Let $x(t, c, \epsilon)$ denote the solution of (4) such that

$$x(0, c, \epsilon) = c$$

where c is an n-vector.

Hopf Bifurcation Theorem. *Suppose that the matrix $A(\epsilon)$ has the eigenvalues*

$$\alpha(\epsilon) + i\beta(\epsilon), \qquad \alpha(\epsilon) - i\beta(\epsilon),$$

where $\alpha(\epsilon)$ is a differentiable function such that $\alpha(0) = 0$, $\alpha'(0) \neq 0$, $\beta(\epsilon)$ is a differentiable function such that $\beta(0) \neq 0$ and where $A(0)$ is such that $i\beta(0)$ is an eigenvalue of multiplicity one and $A(0)$ has no eigenvalue of the form $iq\beta(0)$ where $q = 0, \pm 2, \pm 3, \ldots$. Then there is an interval $I = (-r, r)$, where $r > 0$, and there exist real-valued differentiable functions $\epsilon(s)$, $h(s)$, $c_3(s), \ldots, c_n(s)$, all with domain I, such that

$$\epsilon(0) = h(0) = c_3(0) = \cdots = c_n(0) = 0$$

and such that if

$$c(s) = [0, s, c_3(s), \ldots, c_n(s)],$$

then the solution

$$x[t, c(s), \epsilon(s)]$$

of (4) has period $(2\pi/\beta(0))[1 + h(s)]$.

Proof. For simplicity of notation, we assume that $\beta(0) = 1$ and we choose a coordinate system so that

$$A(0) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \\ 0 & & D \end{bmatrix}$$

where D is an $(n-2) \times (n-2)$ matrix with no eigenvalues of the form iq where $q = 0, \pm 1, \pm 2, \ldots$. Then the equation

$$x' = A(0)x$$

has nontrivial solutions of period 2π and we seek solutions of (4) which have period $2\pi(1+h)$ where h is near zero. A necessary and sufficient condition that the solution $x(t, c, \epsilon)$ have period $2\pi(1+h)$ is:

$$x[2\pi(1+h), c, \epsilon] - x[0, c, \epsilon] = 0. \tag{5}$$

We follow the classical method of Poincaré and search for a vector c and numbers h and ϵ such that (5) is satisfied. By the variation of constants formula,

$$x[2\pi(1+h), c, \epsilon] = e^{2\pi(1+h)A(\epsilon)}c + \int_0^{2\pi(1+h)} e^{[2\pi(1+h)-\sigma]A(\epsilon)} f[x(\sigma, c, \epsilon), \epsilon]\, d\sigma$$

and hence (5) may be rewritten as:

$$[e^{2\pi(1+h)A(\epsilon)} - I]\, c + \int_0^{2\pi(1+h)} e^{[2\pi(1+h)-\sigma]A(\epsilon)} f[x(\sigma, c, \epsilon), \epsilon]\, d\sigma = 0. \tag{6}$$

Since the periodic solution we seek is a solution of an autonomous differential equation and is near the equilibrium point 0, this suggests that we assume that one of the components of the initial value of the solution should be zero. This assumption is indeed completely justified in the proof of the classical result already referred to (Coddington and Levinson [2, pp. 352–353]). Since the solution sought approaches the equilibrium point 0 when $\epsilon \to 0$, it is reasonable to assume that each component of $c(s)$ contains a factor s. Finally a closer study of equation (6) suggests that all components of $c(s)$, except the first two, should be higher order in s. These arguments suggest that we search for an initial condition of the form

$$c = (0, s, sc_3, \ldots, sc_n). \tag{7}$$

Substituting from (7) into equation (6) thus reduces the problem of finding a periodic solution to the problem of solving (6) for $\epsilon, h, c_3, \ldots, c_n$ as functions of s. Next we define the function $F(s, \epsilon, h, c_3, \ldots, c_n)$ as follows:

If $s \neq 0$,

$$F(s, \epsilon, h, c_3, \ldots, c_n) = \left(e^{2\pi(1+h)A(\epsilon)} - I\right)\frac{c}{s} + \frac{1}{s}\int_0^{2\pi(1+h)} e^{[2\pi(1+h)-\sigma]A(\epsilon)} f[x(\sigma, c, \epsilon), \epsilon]\, d\sigma.$$

If $s = 0$,

$$F(0, \epsilon, h, c_3, \ldots, c_n) = \left(e^{2\pi(1+h)A(\epsilon)} - I\right) V$$

where $V = (0, 1, c_3, \ldots, c_n)$. Let $F_1, \ldots, F_n$ denote the components of F.

The equation

$$F(s, \epsilon, h, c_3, \ldots, c_n) = 0 \tag{8}$$

is essentially equation (6) divided by s and consequently the proof of the theorem is complete if we solve (8) for $\epsilon, h, c_3, \ldots, c_n$ as functions of s. By using the fact that

$$f(x, \epsilon) = o(|x|)$$

uniformly in ϵ, it is easy to show that F is continuous and differentiable in a neighborhood of $(0, 0, 0, \ldots, 0)$. By inspection, it is clear that

$$s = \epsilon = h = c_3 = \cdots = c_n = 0$$

is a solution of equation (8). Moreover, a straight-forward computation, using the fact that $\alpha'(0) \neq 0$, shows that the Jacobian

$$JF = \det \begin{bmatrix} \dfrac{\partial F_1}{\partial \epsilon} & \dfrac{\partial F_1}{\partial h} & \dfrac{\partial F_1}{\partial c_3} & \cdots & \dfrac{\partial F_1}{\partial c_n} \\ & & \cdots & & \\ \dfrac{\partial F_n}{\partial \epsilon} & \dfrac{\partial F_n}{\partial h} & \dfrac{\partial F_n}{\partial c_3} & \cdots & \dfrac{\partial F_n}{\partial c_n} \end{bmatrix}$$

where each element in the matrix is evaluated at

$$s = \epsilon = h = c_3 = \cdots = c_n = 0,$$

is nonzero. Hence the implicit function theorem can be applied to (8) and the proof of the theorem is complete.

Now we extend the Hopf Bifurcation Theorem by reducing the hypotheses on the eigenvalues of $A(\epsilon)$. Actually we will show how to apply the bifurcation theorem in [3] to a specific case, i.e., we consider the case in which

$$A(0) = \begin{bmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & D \end{bmatrix} \tag{9}$$

where D is a matrix which has no eigenvalues of the form qi where $q = 0, \pm 1, \pm 2, \ldots$. It will be clear from the description of the procedure how it can be applied to more complicated cases. The theorem from [3] which we want to use is:

BIFURCATION THEOREM. *Suppose U is a bounded open set in R^n and I is the interval $(-\mu_0, \mu_0)$, where $\mu_0 > 0$, on the real line. Let G be an open set in R^n such that $\bar{U} \subset G$ and let $f(x, \mu)$ be a differentiable mapping from $G \times I$ into R^n. Assume that the following conditions are satisfied:*

(1) *There exists $p_0 \in \partial U$ such that*

$$f(p_0, 0) = 0.$$

(2) *If $p \in \partial U - \{p_0\}$, then $f(p, 0) \neq 0$.*

(3) *There exists a neighborhood N in R^n of p_0 such that for each fixed μ with $|\mu|$ sufficiently small, $f[N \cap \partial U, \mu]$ is a surface Σ_μ which has a tangent hyperplane $\mathscr{H}_\mu$ at $f(p_0, \mu)$ and $\mathscr{H}_\mu$ has normal $\mathscr{N}_\mu$ which is a continuous function of μ.*

(4) *The vector $(\partial f/\partial \mu)(p_0, 0)$ is nonzero and*

$$\left[\frac{\partial f}{\partial \mu}(p_0, 0)\right] \cdot \mathscr{N}_0 \neq 0.$$

Conclusion. There exists $\mu_1 > 0$ such that if

$$0 < |\mu| < \mu_1$$

then the Brouwer degree

$$\deg[f(\cdot, \mu), \bar{U}, 0] \tag{i}$$

is defined. For all $\mu \in (0, \mu_1)$ the expression (i) *has the same value (denote it by $d(+)$) and for all $\mu \in (-\mu_1, 0)$, expression* (i) *has the same value (denote it by $d(-)$), and*

$$|d(+) - d(-)| = 1.$$

In order to study the case in which $A(0)$ is given by (9), we look again at the equation

$$F(s, \epsilon, h, c_3, \ldots, c_n) = 0 \tag{8}$$

which was introduced in the proof of the Hopf Bifurcation Theorem. As before, equation (8) has the solution

$$s = \epsilon = h = c_3 = \cdots = c_n = 0.$$

But the implicit function theorem cannot be applied now because at $s = \epsilon = h = c_3 = \cdots = c_n = 0$ the vector $\partial F/\partial c_3$ is zero. Instead we apply the Bifurcation Theorem stated above by regarding $(\epsilon, h, c_3, \ldots, c_n)$ as a point in R^n and letting s be regarded as the parameter μ. In order to apply the Bifurcation Theorem, we must impose conditions on equation (4) so that conditions 1, 2, 3, 4 in the Bifurcation Theorem are satisfied. First we take

$$p_0 = (0, 0, 0, \ldots, 0).$$

The vector $(\partial f/\partial \mu)(0, 0)$ is, in this case, the vector

$$\frac{\partial F}{\partial s}(s, \epsilon, h, c_3, \ldots, c_n)$$

evaluated at $s = \epsilon = h = c_3 = \cdots = c_n = 0$. But since the function f in equation (4) has the property

$$|f(x, 0)| = o(|x|) \tag{10}$$

it is easy to compute $(\partial F/\partial s)(0, 0, \ldots, 0)$ as follows. From definition,

$$\begin{aligned}\frac{\partial F}{\partial s}(0, 0, \ldots, 0) &= \lim_{s\to 0} \frac{F(s, 0, \ldots, 0) - F(0, 0, \ldots, 0)}{s} \\ &= \lim_{s\to 0} \frac{1}{s^2} \int_0^{2\pi} e^{(2\pi-\sigma)A(0)} f[x(\sigma, c, 0), 0]\, d\sigma\end{aligned}$$

where $c = (0, s, 0, \ldots, 0)$. Thus

$$x(\sigma, c, 0) = (x_1(\sigma, c, 0), x_2(\sigma, c, 0), \ldots, x_n(\sigma, c, n))$$

where

$$\begin{aligned}x_2(t, c, 0) &= s + o(|s|)\\ x_j(t, c, 0) &= o(|s|), \qquad j = 1, 3, \ldots, n.\end{aligned}$$

and $o(|s|)$ is uniform in t in the interval $[0, 2\pi]$. From these remarks, it follows that

$$\frac{\partial F}{\partial s}(0, \ldots, 0) = \int_0^{2\pi} e^{(2\pi-\sigma)A(0)} f(I_2, 0)\, d\sigma$$

where

$$I_2 = (0, 1, 0, \ldots, 0).$$

Now we impose the suitable hypotheses on $A(\epsilon)$. We assume that

$$A(\epsilon) = \begin{bmatrix} \alpha(\epsilon) & \beta(\epsilon) & 0 & \\ -\beta(\epsilon) & \alpha(\epsilon) & 0 & \\ 0 & 0 & \gamma(\epsilon) & \\ & & & D(\epsilon)\end{bmatrix}$$

where $\alpha(\epsilon)$, $\beta(\epsilon)$, $\gamma(\epsilon)$ are differentiable functions such that

$$\begin{aligned} &\beta(0) = 1, \\ &\alpha(0) = \gamma(0) = 0, \\ &\alpha'(0) \neq 0, \qquad \beta'(0) \neq -1, \\ &\gamma'(0) \neq 0. \end{aligned}$$

The matrix $D(\epsilon)$ is a differentiable $(n-2) \times (n-2)$ matrix such that $D(0)$ has no eigenvalue of the form $\pm qi$ ($q = 0, 1, 2,...$), and all other entries of $A(\epsilon)$ are identically zero. Since

$$F(0, \epsilon, h, c_3, c_4, ..., c_n) = [e^{2\pi(1+h)A(\epsilon)} - I] \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix}$$

then by Taylor's Expansion, we have:

$$\begin{aligned} &F(0, \epsilon, h, c_3, c_4, ..., c_n) \\ &\quad = [2\pi A'(0)\, e^{2\pi A(0)}\epsilon + 2\pi A(0)\, e^{2\pi A(0)} h] \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix} + H(\epsilon, h) \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix} \end{aligned}$$

where $H(\epsilon, h)$ is an $n \times n$ matrix all of whose entries a_{ij} are second-order in ϵ, h, i.e.,

$$\lim_{|\epsilon|+|h|\to 0} \frac{|a_{ij}(\epsilon, h)|}{|\epsilon| + |h|} = 0,$$

and

$$H(\epsilon, h) = \begin{bmatrix} a_1 & a_{12} & & \\ a_{21} & a_{22} & & \\ & & a_{33} & \\ & & & \mathcal{A}_{n-2} \end{bmatrix} \tag{11}$$

where $\mathcal{A}_{n-2}$ is a matrix and all the entries not written are zero. Hence

$$\begin{aligned} &F(0, \epsilon, h, c_3, c_4, ..., c_n) \\ &\quad = \left\{ 2\pi \begin{bmatrix} \alpha'(0) & \beta'(0) & & \\ -\beta'(0) & \alpha'(0) & & \\ & & \gamma'(0) & \\ & & & D'(0) \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi D(0)} \end{bmatrix} \epsilon \right. \end{aligned}$$

$$+ 2\pi \begin{bmatrix} \alpha(0) & \beta(0) & & \\ -\beta(0) & \alpha(0) & & \\ & & \gamma(0) & \\ & & & D(0) \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi D(0)} \end{bmatrix} h \left\{ \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix} \right.$$

$$+ H(\epsilon, h) \begin{bmatrix} 0 \\ 1 \\ c_4 \\ c_4 \\ \vdots \\ c_n \end{bmatrix}$$

$F(0, \epsilon, h, c_3, c_4, \ldots, c_n)$

$$= \left\{ 2\pi\epsilon \begin{bmatrix} \alpha'(0) & \beta'(0) & & \\ -\beta'(0) & \alpha'(0) & & \\ & & \gamma'(0) & \\ & & & D'(0)\, e^{2\pi D(0)} \end{bmatrix} \right.$$

$$\left. + 2\pi h \begin{bmatrix} \alpha(0) & \beta(0) & & \\ -\beta(0) & \alpha(0) & & \\ & & \gamma(0) & \\ & & & D(0)\, e^{2\pi D(0)} \end{bmatrix} \right\} \begin{bmatrix} 0 \\ 1 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} + H(\epsilon, h) \begin{bmatrix} 0 \\ 1 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

$$= \begin{bmatrix} 2\pi\beta'(0)\,\epsilon + 2\pi\beta(0)\,h \\ 2\pi\alpha'(0)\,\epsilon + 2\pi\alpha(0)\,h \\ 2\pi\epsilon\gamma'(0) + 2\pi h\gamma(0)\,c_3 \\ [e^{2\pi D(0)} - I] \begin{bmatrix} c_4 \\ \vdots \\ c_n \end{bmatrix} \end{bmatrix} + H(\epsilon, h) \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix}. \qquad (12)$$

Note that the first two components of

$$H(\epsilon, h) \begin{bmatrix} 0 \\ 1 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix}$$

are independent of $c_3, c_4, \ldots, c_n$. They are second-order terms in (ϵ, h). Because of the conditions on $\alpha(0)$, $\alpha'(0)$, $\beta(0)$, $\beta'(0)$, the first two components of equation

(12) show that any point in a sufficiently small neighborhood N_0 of the origin which goes into the origin under the mapping

$$(\epsilon, h, c_3, c_4, \ldots, c_n) \to F(0, \epsilon, h, c_3, c_4, \ldots, c_n)$$

must be such that $\epsilon = 0$, $h = 0$. Hence if the set U is chosen so that no point in $\bar{U}$ except the origin is such that $\epsilon = h = 0$, then Conditions 1 and 2 of the Bifurcation Theorem are satisfied. We choose $\bar{U}$ as follows. Let k be a fixed positive number and define the set $\bar{U}$ as:

$$\begin{aligned}\bar{U} = \{(\epsilon, h, c_3, \ldots, c_n)/0 \leqslant \epsilon \leqslant r_1 ; -r_2 \leqslant h \leqslant r_2 ;\\ -k\epsilon \leqslant c_3 \leqslant k\epsilon;\\ 0 \leqslant c_4^2 + \cdots + c_n^2 \leqslant r_3\}\end{aligned}$$

where the positive numbers r_1, r_2, r_3 are chosen small enough so that $\bar{U} \subset N_0$. Then if s is taken to be the parameter μ, the surface $\Sigma_0 = F(\partial U \cap N)$ with $s = 0$, is described in this way: if $p \in \partial U$, then $p = (\epsilon, h, k\epsilon, c_4, \ldots, c_n)$ if the third component of p is nonnegative and $p = (\epsilon, h, -k\epsilon, c_4, \ldots, c_n)$ if the third component of p is negative. Hence if the third component of p is nonnegative, then

$$F(p) = \begin{bmatrix} 2\pi h + 2\pi\beta'(0)\,\epsilon \\ 2\pi\alpha'(0)\,\epsilon \\ 2\pi\gamma'(0)\,k\epsilon^2 \\ [e^{2\pi D(0)} - I]\begin{bmatrix} c_4 \\ \vdots \\ c_n \end{bmatrix} \end{bmatrix} + H(\epsilon, h)\begin{bmatrix} 0 \\ 1 \\ k\epsilon \\ c_4 \\ \vdots \\ c_n \end{bmatrix}$$

and if the third component of p is negative, then

$$F(p) = \begin{bmatrix} 2\pi h + 2\pi\beta'(0)\,\epsilon \\ 2\pi\alpha'(0)\,\epsilon \\ -2\pi\gamma'(0)\,k\epsilon^2 \\ [e^{2\pi D(0)} - I]\begin{bmatrix} c_4 \\ \vdots \\ c_n \end{bmatrix} \end{bmatrix} + H(\epsilon, h)\begin{bmatrix} 0 \\ 1 \\ -k\epsilon \\ c_4 \\ \vdots \\ c_n \end{bmatrix}.$$

Denote $F(p)$ by $\mathscr{F}(\epsilon, h, c_4, \ldots, c_n)$. Then to show that Σ is smooth it is sufficient to show that the matrix

$$M = \left[\frac{\partial\mathscr{F}}{\partial\epsilon} \quad \frac{\partial\mathscr{F}}{\partial h} \quad \frac{\partial\mathscr{F}}{\partial c_4} \quad \cdots \quad \frac{\partial\mathscr{F}}{\partial c_n}\right],$$

where all the derivatives are evaluated at $\epsilon = h = c_4 = \cdots = c_n = 0$, has rank $n - 1$. (Note that since F is a differentiable function of s, this implies Condition 3 of the Bifurcation Theorem.) But a straightforward computation shows that

$$M = \begin{bmatrix} 2\pi\beta'(0) & 2\pi & & & \\ 2\pi\alpha'(0) & 0 & & & \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & [e^{2\pi D(0)} - I] & \\ 0 & 0 & & & \end{bmatrix}.$$

If $\lambda_1, \ldots, \lambda_{n-2}$ are the eigenvalues of $D(0)$, the eigenvalues of $e^{2\pi D(0)}$ are $e^{2\pi\lambda_1}, \ldots, e^{2\pi\lambda_n}$. But $e^{2\pi\lambda_j} = 1$ iff $\lambda_j = \pm qi$, and this last equality is excluded by hypothesis. Hence the matrix $e^{2\pi D(0)} - I$ is nonsingular, and since $\alpha'(0) \neq 0$, it follows that M has rank $n - 1$. Hence Σ is smooth in a neighborhood of the origin and a normal vector to Σ at $F(0)$ is the vector

$$[0, 0, 4\pi^2\alpha'(0) \det[e^{2\pi D(0)} - I], 0, \ldots, 0].$$

Thus Condition 3 of the Bifurcation Theorem is satisfied. And if the third components of $(\partial F/\partial s)(0, \ldots, 0)$ is nonzero, Condition 4 of the Bifurcation Theorem is satisfied. From the Bifurcation Theorem, we obtain thus the following result.

THEOREM 1. *Let the equation*

$$x' = A(\epsilon)x + f(x, \epsilon) \tag{4}$$

satisfy the following conditions:

(1) *$A(\epsilon)$ is a differentiable matrix and*

$$A(\epsilon) = \begin{bmatrix} \alpha(\epsilon) & \beta(\epsilon) & & \\ -\beta(\epsilon) & \alpha(\epsilon) & & \\ & & \gamma(\epsilon) & \\ & & & D(\epsilon) \end{bmatrix}$$

where $\beta(0) = 1$, $\alpha(0) = \gamma(0) = 0$, $\alpha'(0) \neq 0$, $\beta'(0) \neq -1$, $\gamma'(0) \neq 0$ *and* $D(0)$ *has no eigenvalues of the form* $\pm qi$ *where* $q = 0, 1, 2, \ldots$.

(2) *$f(x, \epsilon)$ is a differentiable n-vector function of (x, ϵ) such that:*

(i) $|f(x, \epsilon)| = o(|x|)$ *uniformly in* ϵ.

(ii) *The third component of the vector*

$$\int_0^{2\pi} e^{(2\pi - \sigma)A(0)} f(I_2, 0)\, d\sigma,$$

where

$$I_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

is nonzero.

Conclusion. There exists a closed interval J, such that $J = [0, a]$ where $a > 0$ or $J = [-a, 0]$ where $a > 0$, and a neighborhood N in R^n of

$$(\epsilon, h, c_3, c_4, ..., c_n) = (0, 0, 0, 0, ..., 0)$$

such that for each $s \in J$, there is a point

$$(\epsilon(s), h(s), sc_3(s), sc_4(s), ..., sc_n(s)) \in N$$

such that the solution of (4):

$$x[t, c(s), \epsilon(s)],$$

where

$$c(s) = (0, s, sc_3(s), sc_4(s), ..., sc_n(s));$$

is a nontrivial solution of period $2\pi [1 + h(s)]$.

Remarks. (1) The periodic solution is nontrivial, i.e., it is not simply an equilibrium point; this follows from the hypothesis, $(\partial f/\partial \mu)(p_0, 0) \neq 0$, in the Bifurcation Theorem.

(2) It can be seen from the proof that Theorem 1 is a typical result rather than a general or all-encompassing result. Similar results may be obtained for cases in which the null space of $e^{2\pi A(0)} - I$ has dimension greater than three or if a different set $\bar{U}$ is chosen.

(3) Note that the point $(\epsilon(s), h(s), sc_3(c), sc_4(s), ..., sc_n(s))$ is in the set U and hence that $\epsilon(s) > 0$.

4. An Extension of the Coddington-Levinson Theorem

Now we return to equation (3) and study the case in which the function $g(\epsilon)$ is not identically zero. For $\epsilon = 0$, the point $x = 0$ is an equilibrium point of

$$x' = A(\epsilon)x + f(x, \epsilon) + g(\epsilon) \tag{3}$$

and we are concerned with the question of periodic solutions of (3) which bifurcate from $x = 0$. Hence we seek solutions of (3) of the form

$$x(t, \epsilon) = \epsilon u(t, \epsilon)$$

where $u(t, \epsilon)$ is periodic. Substituting this form into (3) yields:

$$\epsilon u' = \epsilon A(\epsilon)u + f(\epsilon u, \epsilon) + g(\epsilon)$$

or

$$\epsilon u' = \epsilon A(\epsilon)u + f(\epsilon u, \epsilon) - f(0, \epsilon) + g(\epsilon). \tag{13}$$

Using a Taylor's expansion and the fact that

$$|f(x, \epsilon)| = o(|x|)$$

uniformly in ϵ, we may write (13) as:

$$\epsilon u' = \epsilon A(\epsilon)u + \epsilon^2 F(u, \epsilon) + g(\epsilon) \tag{14}$$

where $A(\epsilon)$ is a differentiable function of ϵ and F is a differentiable function of (u, ϵ). Since $g(0) = 0$, then

$$g(\epsilon) = \epsilon h(\epsilon) \tag{15}$$

where $h(\epsilon)$ is differentiable in a neighborhood of $\epsilon = 0$. Substituting from (15) into (14) and dividing by ϵ, we have:

$$u' = A(\epsilon)u + \epsilon f(u, \epsilon) + h(\epsilon)$$

or

$$u' = A(0)u + h(0) + \epsilon G(u, \epsilon) \tag{16}$$

where G is a differentiable function. Without much loss of generality, we may assume that $h(0) = 0$. (If $h(0) \neq 0$, we may translate the coordinate system to eliminate $h(0)$ unless $A(0)$ has an eigenvalue equal to zero. If $A(0)$ has an eigenvalue equal to zero and if P is the projection of R^n on the null space of $A(0)$, we must assume that $P[h(0)] = 0$ in order to effect the translation of the coordinate system.) Rewriting (16) in more familiar notation, we may now state our problem as: does the equation

$$x' = Ax + \epsilon G(x, \epsilon), \tag{17}$$

where A is a constant matrix and $G(x, \epsilon)$ is differentiable, have periodic solutions if $\epsilon \neq 0$? *Stated a little more precisely, we try to determine those periodic* solutions of the equation

$$x' = Ax \tag{18}$$

from which there bifurcate or branch periodic solutions of equation (17) when $\epsilon \neq 0$. This problem has been treated at length by Coddington and Levinson [2] and we will describe an extension of their results. Following Coddington and Levinson, we assume that (18) has nontrivial periodic solutions or, equivalently, that matrix A has pure imaginary eigenvalues. Without restriction on generality, we may assume that A has eigenvalue $\pm i$ and hence that (18) has nontrivial solutions of period 2π. As in Section 3, we start from the classical periodicity condition imposed by Poincaré. The essential difference between the treatments in Sections 3 and 4 is that in Section 3, we introduce a new independent variable s whereas in Section 4, no new independent variable is introduced. Instead the parameter ϵ plays the role of the parameter μ. Let $x(t, c, \epsilon)$ denote the solution of (17) such that $x(0, c, \epsilon) = c$. We seek solutions $x(t, c, \epsilon)$ which have period $2\pi(1 + \epsilon h)$ where h is a function of ϵ which is to be determined. First we impose the usual periodicity condition on solution $x(t, c, \epsilon)$, i.e.,

$$x[2\pi(1 + \epsilon h), c, \epsilon] - x(0, c, \epsilon) = 0. \tag{19}$$

By use of the variation of constants formula, equation (19) may be rewritten as:

$$[e^{2\pi(1+\epsilon h)A} - I]\, c + \int_0^{2\pi(1+\epsilon h)} e^{[2\pi(1+\epsilon h)-\sigma]A} \epsilon G[x(\sigma, c, \epsilon), \epsilon]\, d\sigma = 0. \tag{20}$$

Let $c_1, c_2, \ldots, c_n$ be the components of the vector c. As in Coddington and Levinson [2], we assume that $c_1 = 0$. Thus the problem of finding periodic solutions may be regarded as the problem of solving (20) for $h, c_2, \ldots, c_n$ as functions of ϵ. Equation (20) can be analyzed by using the implicit function theorem or topological degree. Such analysis yields periodic solutions for both positive and negative values of ϵ. The chief drawback to the use of the implicit function theorem is that hypotheses must be imposed which may be practically impossible to verify in many cases. The chief drawback to the use of topological degree is that it requires the computation of a degree and such computation may also be practically impossible. Here we avoid these difficulties by using the Bifurcation Theorem stated in Section 3. However unlike the results derived by using the implicit function theorem or degree, we obtain periodic solutions only for positive ϵ or only for negative ϵ.

The general analysis of equations (20) is bulky and awkward because the equation itself is complicated. However the underlying ideas of the analysis are quite straightforward and instead of a general analysis, we will illustrate it by studying a special case of (20). We assume that A has the form:

$$A = \begin{bmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & D \end{bmatrix}$$

where the unwritten entries are all zeros and D is an $(n - 3) \times (n - 3)$ matrix with no eigenvalues of the form qi ($q = 0, \pm 1, \pm 2, \ldots$). It will be convenient to use the notation

$$E = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we may write:

$$A = \begin{bmatrix} E & \\ & D \end{bmatrix}$$

$$e^{2\pi A} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi D} \end{bmatrix}$$

$$e^{2\pi(1+\epsilon h)A} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{2\pi D} \end{bmatrix} \begin{bmatrix} e^{2\pi\epsilon h\epsilon} & \\ & e^{2\pi\epsilon h D} \end{bmatrix}$$

$$= \begin{bmatrix} e^{2\pi\epsilon h E} & \\ & e^{2\pi(1+\epsilon h)D} \end{bmatrix}$$

$$e^{2\pi(1+\epsilon h)A} - I = \begin{bmatrix} 2\pi\epsilon h E + \dfrac{(2\pi\epsilon h E)^2}{2!} + \cdots & \\ & e^{2\pi(1+\epsilon h)D} - I \end{bmatrix}.$$

Hence equation (20) becomes:

$$\begin{bmatrix} 2\pi\epsilon h E + \dfrac{(2\pi\epsilon h E)^2}{2!} + \cdots & \\ & 2\pi(1+\epsilon h)\,D + \dfrac{[2\pi(1+\epsilon h)\,D]^2}{2!} \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \\ c_3 \\ c_4 \\ \vdots \\ c_n \end{bmatrix} \tag{21}$$
$$+ \int_0^{2\pi(1+\epsilon h)} e^{[2\pi(1+\epsilon h)-\sigma]A} \epsilon G[x(\sigma, c, \epsilon), \epsilon]\, d\sigma = 0.$$

First we solve the last $(n-3)$ component equations in (21) for $c_4, c_5, \ldots, c_n$ in terms of h, c_2, c_3 and ϵ. Let h, c_2, c_3 have arbitrary fixed values. Then the last $(n-3)$ equations have the initial solution

$$\epsilon = c_4 = c_5 = \cdots = c_n = 0.$$

In order to be able to solve the equations uniquely near this initial solution, we apply the implicit function theorem. To use the implicit function theorem we must show that

$$\det[e^{2\pi D} - I] \neq 0.$$

But this follows at once from the hypothesis that matrix D has no eigenvalue of the form qi ($q = 0, \pm 1, \pm 2, \ldots$). Hence we obtain the solutions

$$c_j = c_j(\epsilon), \qquad j = 4, 5, \ldots, n$$

where each $c_j(\epsilon)$ is a differentiable function ($j = 4,\ldots, n$). Now we substitute $c_4(\epsilon),\ldots, c_n(\epsilon)$ into the first three component equations of (21). Then in order to solve equation (20), it is sufficient to solve the three resulting equations for h, c_2, c_3 in terms of ϵ. Let P_3 be the projection

$$P_3\colon R^n \to R^3$$

described by

$$P_3 = (x_1, x_2, \ldots, x_n) \to (x_1, x_2, x_3).$$

Then the three equations may written:

$$\begin{aligned}\left[2\pi\epsilon hE + \frac{(2\pi\epsilon hE)^2}{2!} + \cdots\right] \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} \\ + \epsilon P_3 \int_0^{2\pi(1+\epsilon h)} e^{[2\pi(1+\epsilon h)-\sigma]A} G[x(\sigma, c, \epsilon), \epsilon]\, d\sigma = 0\end{aligned} \tag{22}$$

where

$$c = [0, c_2, c_3, c_4(\epsilon),\ldots, c_n(\epsilon)].$$

Dividing (22) by ϵ and rearranging terms yields:

$$[2\pi hE + \epsilon H(\epsilon, h)] \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} + \int_0^{2\pi(1+\epsilon h)} e^{[2\pi(1+\epsilon h-\sigma)]E} P_3 G[x(\sigma, c, \epsilon), \epsilon]\, d\sigma = 0 \tag{23}$$

where $H(\epsilon, h) = h^2 R(\epsilon, h)$ and R is differentiable for all (ϵ, h).

Now we regard the left side of (23) as a mapping $\mathscr{M}$ from R^4 into R^3, i.e.,

$$\mathscr{M}\colon (h, c_2, c_3, \epsilon) \to R^3.$$

Let M_ϵ denote the mapping from R^3 into R^3 that is obtained by fixing ϵ. We apply the Bifurcation Theorem of Section 3 to the mapping M_ϵ where ϵ plays the role of the parameter μ in the Bifurcation Theorem. Writing the components of (23) more explicitly, we have:

$$\begin{aligned}\begin{bmatrix} 2\pi hc_2 \\ 0 \\ 0 \end{bmatrix} + \epsilon[H(\epsilon, h)] \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} \\ + \int_0^{2\pi(1+\epsilon h)} e^{2\pi(1+\epsilon h)E} \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_1[x(\sigma, c, \epsilon), \epsilon] \\ G_2[x(\sigma, c, \epsilon), \epsilon] \\ G_3[x(\sigma, c, \epsilon), \epsilon] \end{bmatrix} d\sigma = 0\end{aligned}$$

and thus

$$M_0(h, c_2, c_3) = \begin{bmatrix} 2\pi h c_2 \\ 0 \\ 0 \end{bmatrix} + \int_0^{2\pi} \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} G_1[x(\sigma, c, 0), 0] \\ G_2[x(\sigma, c, 0), 0] \\ G_3[x(\sigma, c, d), 0] \end{bmatrix} d\sigma \tag{24}$$

where $c = (0, c_2, c_3, 0, \ldots, 0)$ and

$$x(\sigma, c, 0) = e^{\sigma A} c = \begin{bmatrix} e^{\sigma E} & \\ & e^{\sigma D} \end{bmatrix} \begin{bmatrix} 0 \\ c_2 \\ c_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} c_2 \sin\sigma \\ c_2 \cos\sigma \\ c_3 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

A little more explicitly,

$$M_0(h, c_2, c_3) = \begin{bmatrix} 2\pi h c_2 \\ 0 \\ 0 \end{bmatrix} + \int_0^{2\pi} \begin{bmatrix} (\cos\sigma)\, G_1(c_2 \sin\sigma, c_2\cos\sigma, c_3, 0, \ldots, 0) \\ \qquad - (\sin\sigma)\, G_2(\qquad) \\ (\sin\sigma)\, G_1(\qquad) + (\cos\sigma)\, G_2(\qquad) \\ G_3(\qquad) \end{bmatrix}.$$

Clearly,

$$M_0(0, 0, 0) = 0.$$

Now suppose that

$$\begin{aligned} G_1(\xi_1, \xi_2, \xi_3, 0, \ldots, 0) &= \alpha_1 + \alpha_{11}\xi_1 + \alpha_{12}\xi_2 + \alpha_{13}\xi_3 + h_1(\xi_1, \xi_2, \xi_3) \\ G_2(\xi_1, \xi_2, \xi_3, 0, \ldots, 0) &= \alpha_2 + \alpha_{21}\xi_1 + \alpha_{22}\xi_2 + \alpha_{23}\xi_3 + h_2(\xi_1, \xi_2, \xi_3) \\ G_3(\xi_1, \xi_2, \xi_3, 0, \ldots, 0) &= \alpha_3 + \alpha_{31}\xi_1 + \alpha_{32}\xi_2 + \alpha_{33}\xi_3 + h_3(\xi_1, \xi_2, \xi_3) \end{aligned}$$

where α_1, α_2 are constants and $h_i(\xi_1, \xi_2, \xi_3)$ is a higher order term, i.e.,

$$\lim_{|(\xi_1, \xi_2, \xi_3)| \to 0} \frac{|h_i(\xi_1, \xi_2, \xi_3)|}{|\xi_1| + |\xi_2| + |\xi_3|} = 0,$$

and assume that $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ and

$$\alpha_{33} \neq 0$$

and

$$\alpha_{12} \neq \alpha_{21}$$

or

$$\alpha_{11} \neq -\alpha_{22}.$$

Then it follows easily that any point (h, c_2, c_3) in a sufficiently small neighborhood N_0 of the origin which goes into the origin under the mapping

$$(h, c_2, c_3) \to M_0(h, c_2, c_3)$$

must be such that $c_2 = c_3 = 0$. Hence if the set U is chosen so that no point in $\bar{U}$ except the origin is such that $c_2 = c_3 = 0$, then conditions 1 and 2 of the Bifurcation Theorem are satisfied. We choose $\bar{U}$ as follows. Let k be a fixed positive number and define the set $\bar{U}$ as:

$$\bar{U} = \{(h, c_2, c_3)/0 \leqslant c_2 \leqslant r_1 ; -r_2 \leqslant c_3 \leqslant r_2 ; -kc_2 \leqslant h \leqslant kc_2\} \tag{25}$$

where the positive numbers r_1, r_2 are chosen small enough so that $\bar{U} \subset N_0$. Then the surface $\Sigma_0 = M_0(\partial U \cap N)$ is described this way. If $p \in \partial U$, then $p = (kc_2, c_2, c_3)$ if the first component of p is nonnegative and $p = (-kc_2, c_2, c_3)$ if the first component of p is negative. Hence if the first component of p is nonnegative then

$$M_0(p) = \begin{bmatrix} 2\pi k c_2{}^2 \\ 0 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} \alpha_{12}c_2 - \alpha_{21}c_2 \\ \alpha_{11}c_2 + \alpha_{22}c_2 \\ 2\alpha_3 c_3 \end{bmatrix} + \mathscr{H}(c_2, c_3)$$

where $\mathscr{H}(c_2, c_3)$ is higher order in (c_2, c_3). If the third component of p is nonnegative then

$$M_0(p) = \begin{bmatrix} -2\pi k c_2{}^2 \\ 0 \\ 0 \end{bmatrix} + \pi \begin{bmatrix} (\alpha_{12} - \alpha_{21})c_2 \\ (\alpha_{11} + \alpha_{22})c_2 \\ +2\alpha_{33}c_3 \end{bmatrix} + \mathscr{H}(c_2, c_3)$$

where $\mathscr{H}(c_2, c_3)$ is higher in (c_2, c_3). Thus to show that Σ_0 is smooth it is sufficient to show that the matrix

$$\begin{bmatrix} \alpha_{12} - \alpha_{21} & 0 \\ \alpha_{11} + \alpha_{22} & 0 \\ 0 & 2\alpha_{33} \end{bmatrix}$$

has rank 2. But this follows at once from the assumptions on $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ and α_{33}. Also we have at once that the normal to Σ_0 at $M_0(0)$ is the vector

$$[(\alpha_{11} + \alpha_{22})\, 2\alpha_{33}, (\alpha_{12} - \alpha_{21})\, 2\alpha_{33}, 0]. \tag{26}$$

But it is a straightforward computation to show that

$$\left.\frac{\partial M_\epsilon}{\partial \epsilon}\right]_{h=c_2=\cdots=c_n=\epsilon=0} = \int_0^{2\pi} \begin{bmatrix} \cos\sigma & -\sin\sigma & 0 \\ \sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\frac{\partial G}{\partial x}\frac{\partial x}{\partial \epsilon} + \frac{\partial G}{\partial \epsilon}\right] d\sigma$$

where

$$\frac{\partial G}{\partial x}\frac{\partial x}{\partial \epsilon} = \begin{bmatrix} \dfrac{\partial G_1}{\partial x_1} & \dfrac{\partial G_1}{\partial x_2} & \dfrac{\partial G_1}{\partial x_3} \\ \dfrac{\partial G_2}{\partial x_1} & \dfrac{\partial G_2}{\partial x_2} & \dfrac{\partial G_2}{\partial x_2} \\ \dfrac{\partial G_3}{\partial x_1} & \dfrac{\partial G_3}{\partial x_3} & \dfrac{\partial G_3}{\partial x_3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial x_1}{\partial \epsilon} \\ \dfrac{\partial x_2}{\partial \epsilon} \\ \dfrac{\partial x_3}{\partial \epsilon} \end{bmatrix}, \qquad \frac{\partial G}{\partial \epsilon} = \begin{bmatrix} \dfrac{\partial G_1}{\partial \epsilon} \\ \dfrac{\partial G_2}{\partial \epsilon} \\ \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}$$

and

$$\frac{\partial G_i}{\partial x_j}, \qquad \frac{\partial G_i}{\partial \epsilon}, \qquad \frac{\partial x_i}{\partial \epsilon},$$

are evaluated at $\epsilon = c_1 = c_2 = \cdots = c_n = 0$. A straight-forward computation shows that

$$\frac{dx}{d\epsilon}(t, 0, 0) = \begin{bmatrix} \alpha_1 \sin t + \alpha_2 - \alpha_2 \cos t \\ -\alpha_1 + \alpha_1 \cos t + \alpha_2 \sin t \\ 0 \end{bmatrix},$$

$$\left.\frac{\partial G}{\partial x}\right]_{\epsilon=c_1=\cdots=c_n=0} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{23} \end{bmatrix},$$

and

$$\int_0^{2\pi} e^{\sigma A}\left\{\left.\frac{\partial G}{\partial \epsilon}\right]_{\epsilon=0, c_i=0, i=1,\ldots,n}\right\} d\sigma = \begin{bmatrix} 0 \\ 0 \\ 2\pi \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}$$

where $\partial G_3/\partial \epsilon$ is evaluated at $\epsilon = c_1 = \cdots = c_n = 0$. Hence

$$\left.\frac{\partial M_\epsilon}{\partial \epsilon}\right]_{h=c_2=\cdots=c_n=\epsilon=0}$$

$$= \int_0^{2\pi} \begin{bmatrix} \cos\sigma & \sin\sigma & 0 \\ -\sin\sigma & \cos\sigma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_1 \sin\sigma + \alpha_2 - \alpha_2 \cos\sigma \\ -\alpha_1 + \alpha_1 \cos\sigma + \alpha_2 \sin\sigma \\ 0 \end{bmatrix} d\sigma + \begin{bmatrix} 0 \\ 0 \\ 2\pi \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}$$

where $\partial G_3/\partial \epsilon$ is evaluated at $\epsilon = c_1 = \cdots = c_n = 0$, and

$$\left.\frac{\partial M_\epsilon}{\partial \epsilon}\right]_{h=c_2=\cdots=c_n=\epsilon=0}$$

$$= \int_0^{2\pi} \begin{bmatrix} \alpha_{11}\cos\sigma + \alpha_{21}\sin\sigma & \alpha_{12}\cos\sigma + \alpha_{22}\sin\sigma & \alpha_{13}\cos\sigma + \alpha_{23}\sin\sigma \\ -\alpha_{11}\sin\sigma + \alpha_{21}\cos\sigma & -\alpha_{12}\sin\sigma + \alpha_{22}\cos\sigma & -\alpha_{13}\sin\sigma + \alpha_{23}\cos\sigma \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

$$\times \begin{bmatrix} \alpha_1 \sin\sigma + \alpha_2 - \alpha_2 \cos\sigma \\ -\alpha_1 + \alpha_1 \cos\sigma + \alpha_2 \sin\sigma \\ 0 \end{bmatrix} d\sigma + \begin{bmatrix} 0 \\ 0 \\ 2\pi \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}$$

$$= \int_0^{2\pi} \begin{bmatrix} -\alpha_{11}\alpha_2 \cos^2 \sigma + \alpha_{21}\alpha_1 \sin^2 \sigma + \alpha_{12}\alpha_1 \cos^2 \sigma + \alpha_{22}\alpha_2 \sin^2 \sigma \\ -\alpha_{11}\alpha_1 \sin^2 \sigma - \alpha_{21}\alpha_2 \cos^2 \sigma - \alpha_{12}\alpha_2 \sin^2 \sigma + \alpha_{22}\alpha_1 \cos^2 \sigma \\ \alpha_{31}\alpha_2 \qquad - \alpha_{32}\alpha_1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 2\pi \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}.$$

Since

$$\int_0^{2\pi} \sin^2 x \, dx = \pi = \int_0^{2\pi} \cos^2 x \, dx,$$

then,

$$\frac{\partial M_\epsilon}{\partial \epsilon}(0) = 2\pi^2 \begin{bmatrix} \alpha_{11}\alpha_2 + \alpha_{21}\alpha_1 + \alpha_{12}\alpha_1 + \alpha_{22}\alpha_2 \\ -\alpha_{11}\alpha_1 - \alpha_{21}\alpha_2 - \alpha_{12}\alpha_2 + \alpha_{22}\alpha_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2\pi \dfrac{\partial G_3}{\partial \epsilon} \end{bmatrix}. \tag{27}$$

Inspection of (26) and (27) shows that conditions 3 and 4 of the Bifurcation Theorem are satisfied if:

$$\begin{aligned} &(\alpha_{11} + \alpha_{22})\, \alpha_{33}[\alpha_{11}\alpha_2 + \alpha_{21}\alpha_1 + \alpha_{12}\alpha_1 + \alpha_{22}\alpha_2] \\ &\quad + (\alpha_{12} - \alpha_{21})\, \alpha_{33}[-a_{11}\alpha_1 - \alpha_{21}\alpha_2 - \alpha_{12}\alpha_2 + \alpha_{22}\alpha_1] \neq 0. \end{aligned} \tag{28}$$

We summarize this discussion in the following theorem.

THEOREM 2. *Let the equation*

$$x' = Ax + \epsilon G(x, \epsilon) \tag{17}$$

satisfy the following conditions:

(1) *The matrix A has the form*

$$A = \begin{bmatrix} 0 & 1 & 0 & \\ -1 & 0 & 0 & \\ 0 & 0 & 0 & \\ & & & D \end{bmatrix}$$

where D is an $(n - 3) \times (n - 3)$ matrix with no eigenvalues of the form qi ($q = 0, \pm 1, \pm 2, \ldots$).

(2) *$G(x, \epsilon)$ is a continuously differentiable n-vector function such that the first three components of $G(x, \epsilon)$ have the form*:

$$G_i(x_1, x_2, x_3, 0, \ldots, 0) = \alpha_i + \alpha_{i1}x_1 + \alpha_{i2}x_2 + \alpha_{i3}x_3 + h_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3),$$

where

$$\lim_{|x_1|+|x_2|+|x_3|\to 0} \frac{|h_i(x_1, x_2, x_3)|}{|x_1| + |x_2| + |x_3|} = 0,$$

and

$$\alpha_1 \neq 0 \quad \textit{or} \quad \alpha_2 \neq 0,$$
$$\alpha_{33} \neq 0,$$
$$\alpha_{12} \neq +\alpha_{21} \quad \textit{or} \quad \alpha_{11} \neq -\alpha_{22}.$$

Also condition (28) *is satisfied.*

Conclusion. There exists a closed interval J such that $J = [0, a]$ where $a > 0$ or $J = [-a, 0]$ where $a > 0$ and a neighborhood N in R^n of

$$(h, c_2, \ldots, c_n) = (0, \ldots, 0)$$

such that for each $\epsilon \in J$ there is a point

$$(h(\epsilon), c_2(\epsilon), \ldots, c_n(\epsilon)) \in N$$

such that the solution of (17)

$$x(t, c(\epsilon), \epsilon),$$

where

$$c(\epsilon) = (0, c_2(\epsilon), \ldots, c_n(\epsilon)),$$

is a nontrivial solution of period $2\pi[1 + \epsilon h(\epsilon)]$.

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